Estimates of VaR for Itô Processes

Begoña Fernández

Facultad de Ciencias, UNAM
PASI, may, 2010, CIMAT

Risk Measures

1. VaR_α and Expected Shortfall.
2. Ruin Probability.
3. Expectation with respect an utility function.

VaR_α and Expected Shortfall

- If X is a r.v. $VaR_\alpha(X)$ is defined by

$$\begin{aligned} VaR_\alpha(X) &= \inf\{z \in R \mid P(X > z) < \alpha\} \\ &= \inf\{z \in R \mid P(X \leq z) \geq q\}, \end{aligned}$$

  $\alpha = 1 - q$.
- Basel Accords regulate the credit institutions. One criteria is VaR_α $1 - \alpha = .99$, where X represent the loss. The institutions must be covered for losses inferior to this quantile.
- Expected Shortfall

$$E[X - VaR\alpha \, |X > VaR_\alpha]$$

Difficulties

Estimation of very small probabilities. Different Methods:

1. Variance Reduction.
2. Bayesian
3. Extreme Value Theory. (Embrechts et al.)

For independent r.v. and for some kinds of dependence

Ruin Probability in Insurance

- Cramér-Lundberg Model:

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i$$

$$\textit{Ruin Probability} = P[X_t < 0, \text{p. a. } t > 0] = \psi(x)$$

- Find x such the ruin probabilty $\Psi(x) \leq q$.

$$\textit{Survival Probability} = \delta(x) = 1 - \psi(x)$$

$$\delta'(x) = \frac{\lambda}{c}\delta(x) - \frac{\lambda}{c}\int_0^x \delta(x-y)f(y)dy$$

If the r.v. . Y_i admit Laplace transform

$$Ce^{-\theta x} \leq \psi(x) \leq e^{-\theta x}$$

Optimization w. r. Utility Function

- Utility function U, represents the risk aversion. (Concave Functions).

$$E[U(X)]$$

Stochastic Processes

If we have a stochastic processes $X_t, \; t \in [0, T]$ that represents the wealth or the price of a finantial asset, what is the random variable we choose to calculate the Var?

- For each $t \in [0, T], \; X_t$. Levy Processes. Important property independent and stationary increments.
- Some kind of Dynamic Var (Mc-Kean, Embrechts et al.) for time Series.

The General Model

Let $B_t,\ t \geq 0$ be a Brownian Motion and $N_t,\ t \geq 0$ a Poisson Process independent of B.M.

$$X_t = x + \int_0^t \sigma(s, \omega X_s) dB_s + \int_0^t b(s, \omega, X_s) ds + \sum_{i=1}^{N_t} \gamma_{T_i^-} Y_i,\ t > 0,$$

1. Estimates of VaR and expected shortfall for processes without jumps.
2. Estimates of VaR and Ruin Probabilities for the General Model.

Joint work with Laurent Denis and Ana Meda.

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \; t > 0, \; X_0 = m \tag{1}$$

We can take as our r.v X_t for $t > 0$ fixed.

- Estimates for the density (when it exists) of X_t can be used to obtain bounds for the VaR and the expected shortfall. In general, these are lower estimates. (Bally, V. Talay, D).

For $t \in R_+$ fixed

$$X_t^* = \sup_{0 \leq s \leq t} X_s$$

$$VaR_{m,\alpha}^t = \inf\{z \in R,\ P_m(X_t^* \geq z) \leq \alpha\}.$$

$$E[X_t^* - VaR_{m,\alpha}^t \,|X_t^* > VaR_{m,\alpha}^t] = E[X_t^* \,|X_t^* > VaR_{m,\alpha}^t] - VaR_{m,\alpha}^t$$

Main Idea

Obtain upper and lower bounds for the tail distribution of X_t^*, under some conditions of the coefficients and then

1. $VaR^t_{m,\alpha}$.
2. Expected shortfall.
3. The toy models.

Toy Models

1. *Geometric Brownian Type*

$$dX_t = b_t X_t dt + \sigma_t X_t dW_t.$$

2. *Cox-Ingersol Ross* (Positive Process):

$$dX_t = (b - cX_t)dt + \sqrt{a^*}\sqrt{X_t}dW_t, \quad b, c, a \in Z^+$$

3. *Vasicek Type Model*

$$dX_t = (\beta - \mu X_t)dt + \sigma dW_t, \quad \mu, \sigma \in R^+, \beta \in R.$$

Hipothesis (UB):

1. $b(t,z) \leq b^*$, $b^* \geq 0$.
2. $|\sigma(t,z)| \leq \sqrt{a^*}(z^+)^\gamma$, $a^* > 0$, $\gamma \in [0,1)$.

1. Geometric Brownian Type Model We will take

$$Y_t = \log(X_t), \quad dY_t = (b_t - \frac{\sigma_t^2}{2})dt + \sigma_t dW_t.$$

2. Cox-Ingersoll-Ross (Positive Process)

$$b(t,z) = b - cz^+ \leq b = b^*, \ \sigma(t,z) = \sqrt{a^*}\sqrt{|z|}$$

3. Vasicek Type Model

$$Y_t = e^{\int_0^t \mu_s ds} X_t, \quad dY_t = \beta e^{\int_0^t \mu_s ds} dt + \sigma e^{\int_0^t \mu_s ds} dW_t$$

Lemma

Assume **(UB)** *then for* $t > 0$, $z \in \mathbb{R}$ *we have*

$$P_m(X_t^* \geq z) \leq 2\bar{\Phi}\left(\frac{(z-m-b^*t)^+}{\sqrt{a^*tz^\gamma}}\right).$$

where

$$\bar{\Phi}(x) = \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-\frac{u^2}{2}}du.$$

Proof

If $z \geq m$, then we can assume

$$\text{For all } y \geq z, \; a(y) = a(z).$$

$$P(X_t^* \geq z) \quad \leq \quad P(\sup_{0 \leq s \leq t} \int_0^s \sigma(X_u) dW_u \geq z - m - b^* t)$$

$$R_s = \int_0^s \sigma(X_u) dW_u, \quad R_s = \tilde{B}_{\langle R,R \rangle_s}, \quad \tilde{B}, \; B.M.$$

$$\begin{aligned} P(X_t^* \geq z) \quad & \leq \quad P(\sup_{0 \leq s \leq t} \tilde{B}_{\langle R,R \rangle_s} \geq z - m - b^* t) \\ & \leq \quad P(\sup_{0 \leq u \leq a^* t z^{2\gamma}} \tilde{B}_u \geq z - m - b^* t) \\ & = \quad 2\bar{\Phi}\left(\frac{(z - m - b^* t)^+}{\sqrt{a^* t z^{\gamma}}} \right). \end{aligned}$$

Upper Estimates

Theorem

Assume **(UB)**. *For each* $0 < t$,

$$VaR^t_{m,\alpha} \leq r,$$

where r *is the unique root in* $[b^*(t) + m, +\infty[$ *of the equation*

$$z - z^\gamma \sqrt{a^*(t)}\bar{\Phi}^{-1}(\alpha/2) - m - b^*(t) = 0.$$

Corollary

Assume (UB)

1. *Geometric Brownian Type. Assume* $b_s \leq b^*, \; a_* \leq \sigma_s \leq a^*$:

$$VaR^t_{m,\alpha} \leq m e^{t(b^* - \frac{a_*}{2})^+ + \sqrt{a^* t}\bar{\Phi}(\frac{\alpha}{2})}$$

2. *CIR*

$$\begin{aligned} VaR^t_{m,\alpha} &\leq m + b^* + \frac{1}{2} a^* (\bar{\Phi}^{-1})^2(\alpha/2) \\ &+ \frac{1}{2}\bar{\Phi}^{-1}(\alpha/2)\sqrt{a^*(\bar{\Phi}^{-1})^2(\alpha/2) + 4(m + b^*)} \end{aligned}$$

3. *Vasicek Type Model* $\mu_t = \mu$

$$VaR^t_{m,\alpha} \leq m + \beta e^{\mu t} + e^{\mu t}\sigma\sqrt{t}\bar{\Phi}^{-1}(\alpha/2)$$

Proposition

Assume (UB)

$$E(X_t^*|X_t^* \geq VaR_{m,\alpha}^t) \leq \frac{2}{\alpha}\int_{VaR_{m,\alpha}^t}^{\infty} \bar{\Phi}\left(\frac{(z-m-b^*)^+}{\sqrt{a^*}z^\gamma}\right) dz$$

Hypothesis LB

For all $z \in R$, $t \geq 0$,

- $b(t, z) \geq b_*$, $b_* \leq 0$.
- $\sigma(t, z) \geq a_*$, $a_* > 0$.

Lemma

Assume (**LB**). *Then for all* $z \in \mathbb{R}$ *y* $t \geq 0$,

$$2\bar{\Phi}\left(\frac{(z - m - b_* t)^+}{\sqrt{a_* t}}\right) \leq P(X_t^* \geq z).$$

$$\bar{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{u^2}{2}} du.$$

Theorem

Assume (**LB**). *Then, for all* $t > 0$,

$$VaR^{t}_{m,\alpha}(X) \geq m + b_* t + \sqrt{a_* t}\,\bar{\Phi}^{-1}(\alpha/2).$$

Corollary

If $\gamma = 0$, *i.e.* σ *bounded*,

$$VaR^{s,t}_{m,\alpha}(X) \leq m + b^*(t-s) + \sqrt{a^*(t-s)}\,\bar{\Phi}^{-1}(\alpha/2).$$

X Geometric Brownian Type.

$$dX_t = b_t X_t dt + a_t X_t dW_t.$$

$$Y_t = \ln(X_t), \qquad \text{for all } t \geq 0$$

$$Y_t = ln(m) + \int_0^t \sigma_s \, dB_s + \int_0^t (b_s - \frac{\sigma_s^2}{2}) ds$$

If there exist constants $0 < a_* \leq a^*$ and $b_* \leq b^*$ such that $t \geq 0$

$$b_* \leq b_t \leq b^* \text{ and } a_* \leq \sigma_t^2 \leq a^*,$$

then for all $z \geq \ln m$ and $t \geq 0$

$$P(X_t^* \geq z) \leq 2\bar{\Phi}\left(\frac{(z - \ln m - t(b^* - \frac{a_*}{2})^+)^+}{\sqrt{a^* t}}\right).$$

$$2\bar{\Phi}\left(\frac{(z - \ln m - t(b_* - \frac{a^*}{2})^-)^+}{\sqrt{a_* t}}\right) \leq P(X_t^* \geq z).$$

Aplication

Consider a Russian option, on a Geometric Brownian type process. Let $t > 0$ be expiration time. The pay-off is given by

$$f_s = M_0 \bigvee \sup_{u \leq s} X_u.$$

If M_0 is fixed, $P_m(X_t^* \geq M_0)$ is the risk for the writter.

We can use the VaR to fix the value M_0.

Dynamic VaR

$$X^*_{s,t} = \sup_{s \le r \le t} X_r$$

We want to measure the VaR of $X^*_{s,t}$ given that the process X exceeded VaR^s_α
This would help to know how risky the future is (up to time $t > s$), given that you already exceeded VaR in the past.
In a natural way we define

$$\widetilde{VaR}^{s,t}_\alpha(X) = \inf\left\{z \in \mathbb{R},\ P_x\left(X^*_{s,t} < z | X^*_s \ge VaR^s_{m,\alpha}\right) \ge q\right\}.$$

Theorem

Assume the process X has the Markov property, then

$$\widetilde{VaR}_{\alpha}^{s,t}(X) \leq r,$$

where r is the unique root on $[b^* t + VaR^s_{m,\alpha}, +\infty[$ *of the following equation:*

$$z - z^{\gamma}\sqrt{a^* t}\,\bar{\Phi}^{-1}(\alpha/2) - VaR^s_{m,\alpha} - b^* t = 0. \tag{2}$$

Proof

$$P_x\left(X^*_{s,t} \geq z | X^*_s \geq VaR^s_{x,\alpha}(X)\right) \leq \frac{P_x\left(X^*_{s,t} \geq z, X^*_s \geq VaR^s_{x,\alpha}(X)\right)}{P_x(X^*_s \geq VaR^s_{x,\alpha}(X))}.$$

We now introduce

$$T = \inf\left\{u > 0,\ X_u = VaR^s_{x,\alpha}(X)\right\},$$

denote by μ the law of T under P_x. Then

$$\begin{aligned} P_x\left(X^*_{s,t} \geq z, X^*_s \geq VaR^s_{x,\alpha}(X)\right) &= \int_0^s P_x\left(X^*_{s,t} \geq z | T = r\right)\mu(dr) \\ &\leq \int_0^s P_x\left(X^*_{r,t} \geq z | T = r\right)\mu(dr) \\ &= \int_0^s P_{VaR^s_{x,\alpha}(X)}\left(X^*_{t-r} \geq z\right)\mu(dr) \end{aligned}$$

$$\int_0^s P_{VaR^s_{x,\alpha}(X)}\left(X^*_{t-r} \geq z\right)\mu(dr)$$

$$\leq \int_0^s 2\bar{\Phi}\left(\frac{(z - VaR^s_{x,\alpha}(X) - b^*(t-r))^+}{\sqrt{a^*(t-r)}z^\gamma}\right)\mu(dr)$$

$$\leq 2\bar{\Phi}\left(\frac{(z - VaR^s_{x,\alpha}(X) - b^*t)^+}{\sqrt{a^*t}z^\gamma}\right)P_x(T \leq s)$$

$$= 2\bar{\Phi}\left(\frac{(z - VaR^s_{x,\alpha}(X) - b^*t)^+}{\sqrt{a^*t}z^\gamma}\right)P_x(X^*_s \geq VaR^s_{x,\alpha}(X)).$$

So,

$$P_x\left(X^*_{s,t} \geq z | X^*_s \geq VaR^s_{x,\alpha}(X)\right) \leq 2\bar{\Phi}\left(\frac{(z - VaR^s_{x,\alpha}(X) - b^*t)^+}{\sqrt{a^*t}z^\gamma}\right),$$

As previously, for $\gamma = 0$ or $\gamma = 1/2$ we are able to calculate this bound and this yields:

Corollary

(i) *If $\gamma = 0$, that is σ bounded, there is the following estimate:*

$$\widetilde{VaR}_{\alpha}^{s,t}(X) \leq VaR_{x,\alpha}^{s}(X) + b^* t + \sqrt{a^* t}\bar{\Phi}^{-1}(\alpha/2).$$

(ii) *If $\gamma = 1/2$ we have:*

$$\widetilde{VaR}_{\alpha}^{s,t}(X) \leq VaR_{x,\alpha}^{s}(X) + b^* t + \frac{1}{2}a^* t(\phi^{-1})^2(\alpha/2)$$
$$+\frac{1}{2}\phi^{-1}(\alpha/2)\sqrt{a^* t\left(a^* t(\phi^{-1})^2(\alpha/2) + 4(VaR_{x,\alpha}^{s}(X) + b^* t\right.}$$

Thanks

- D.G. Aronson: Bounds on the fundamental solution of a parabolic equation. Bull. Am. Math. Soc. 73 (1967), pp. 890-896.
- V. Bally: Lower bounds for the density of Ito processes. To appear in Ann. of Proba.
- V. Bally,M.E.Caballero, N. El-Karoui,B.Fernandez: Reflected BSDE's , PDE's and Variational Inequalities. To appear in Bernoulli.
- A. Bensoussan, J. L. Lions: Applications des Inéquations variationelles en control stochastique. Dunot, Paris, 1978.
- P. Bjerksund, G. Stensland: Closed Form Approximation of American Options, Scandinavian Journal of Mannagement, Vol. 9, Suppl., 1993,pp. 587-599.
- M. Broadie, J. Detemple: American option valuation: new bounds, approximation, and comparison of existing methods. Review of Financial Studies, 1996, Vol. 9, No. 4, pp. 1211-1250.

- L. Denis, B. Fernández, A. Meda Estimates of Daynamic VaR and Mean Loss associated to diffusion processes. IMS collectio. Vol. 4 Markov Processes and Related Topics: A Festchrift for Thomas Kurtz. , 2008, 301-314.
- N. El Karoui, C. Kapoudjan, E. Pardoux, S. Peng, M.C. Quenez: Reflected solutions of Backward SDE's, and related obstacle problems for PDE's. The Annals of Probability,1997, Vol. 25,No 2, 702-737.
- J-P Fouque, G. Papanicolau and K.R. Sircar: Derivatives in financil markets with stochastic volatility. Cambridge University Press, 2000.
- H. Guérin, S. Meleard, E. Nualart: Exponential estimates for spatially homogeneous Landau equation via Malliavin calculus. Preprint PMA 876,(2004).

- A. Kohatsu-Higa: Lower bounds for densities of uniform elliptic random variables on the Wiener space. PTRF 126, pp. 421-457, (2003).
- D. Kinderlehrer and G. Stampacciat: An introduction to variational inequalities and their applications. Academic Press, 1980.
- A. Kohatsu-Higa: Lower bounds for the density of uniform elliptic random varibles on the Wiener space. PTRF, 126, pp. 421-457 (2003).
- Ikeda, S. Watanabe: Stochastic Differential Equations and Diffusion Processes. Amsterdam, Oxford, New York, Notrh Holland, Kodansha, 1989.
- S. Mohamed: Stochastic differential systems with memory: theory, examples and applications; Gelio Workshop 1996, Progress in Probability, vol. 42, 1996.

D. Strook: Diffusion semigroups corresponding to uniform elliptic divergence form operators. Seminaires de Probabilitées XXII, Lecture Notes in Mathematics, 1988.